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Elliptically Polarized Fields and Responses

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An elliptically polarized field applied to a physical system and related responses are very common in physics. Due to the loss of symmetry, the response problems are very difficult to solve, and are usually described by nonlinear and unseparable equations. By introducing a time transformation $\tau = (1/\omega) \tan^{-1}(r \tan \omega t)$, where r is the ratio between the two components, one may reset the symmetry of the field. The equation

$$\frac{d\theta}{dt} = \frac{\omega r}{\cos^2 \omega t + r^2 \sin^2 \omega t} - \omega_c(\cos^2 \omega t + r^2 \sin^2 \omega t) \sin 2\theta$$

describing the relative angular motion of a magnetic dipole pair in an elliptical magnetic field has been solved with this transformation.

It is very common in physics that a system is acted on by an external alternating field. If the external field is elliptically polarized, due to the loss of spacial symmetry, the problem usually becomes much more complicated than for circularly polarized fields. The dynamical equations of the physical system are very difficult to solve. In this note, through introducing a time transformation, we can transfer the elliptically polarized field into a circularly polarized field. Using this time transformation, we solve the dynamical equation of relative motion of a dipole pair in an external elliptically polarized magnetic field.

Consider a physical system in interaction with an elliptically polarized field. Let H_x and H_y represent the x and y components of the polarized field $\mathbf{H}(t)$, which can be expressed by

$$H_x = H_0 \cos \omega t \qquad \text{and} \qquad H_y = rH_0 \sin \omega t \tag{1}$$

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where H_0 and rH_0 are x and y components of the amplitudes, respectively, and r ($0 < r \leq 1$) is the ratio between the amplitudes. The angular velocity of the field is not a constant, but a function of time:

$$\frac{d\varphi_H}{dt} = \omega_H = \frac{\omega r}{\cos^2 \omega t + r^2 \sin^2 \omega t} \tag{2}$$

We now introduce a time transformation:

$$\tau = \frac{1}{\omega} \tan^{-1}(r \tan \omega t) \tag{3}$$

This yields

$$\frac{d\varphi_H}{d\tau} = \omega = \text{const} \tag{4}$$

If we use τ as a time measure, then in τ time, the elliptically polarized field becomes a circularly polarized field.

The relative angular motion of a magnetic dipole pair acted on by an external elliptically polarized magnetic field with frequency ω can be described by the following equation (Helgesen *et al.*, 1990)

$$\frac{d\theta}{dt} = \frac{\omega r}{\cos^2 \omega t + r^2 \sin^2 \omega t} - \omega_c(\cos^2 \omega t + r^2 \sin^2 \omega t) \sin 2\theta \tag{5}$$

where the θ is the angle between the applied elliptically polarized magnetic field and the connecting line of the dipole pair (see Figure 1). The constant ω_c is the critical frequency of the system.

Let

$$u = \tan \theta, \qquad d\theta = \frac{du}{1+u^2}, \qquad \sin 2\theta = \frac{2u}{1+u^2} \tag{6}$$

and using the transformation (3), we have

$$\frac{d\tau}{dt} = \frac{r}{\cos^2 \omega t + r^2 \sin^2 \omega t} = \frac{1 + (1/r^2) \tan^2 \omega\tau}{1 + \tan^2 \omega\tau} r \tag{7}$$

Substituting equations (6) and (7) into equation (5), we have

$$\frac{du}{d\tau} = \omega u^2 - \frac{2\omega_c}{r}\left(\frac{1 + \tan^2 \omega\tau}{1 + (1/r^2) \tan^2 \omega\tau}\right)^2 u + \omega \tag{8}$$

Let $\omega = 1$ (without loss of universality); we have

$$\frac{du}{d\tau} = u^2 - \frac{2\omega_c}{r}\left(\frac{1 + \tan^2 \tau}{1 + (1/r^2) \tan^2 \tau}\right)^2 u + 1 \tag{9}$$

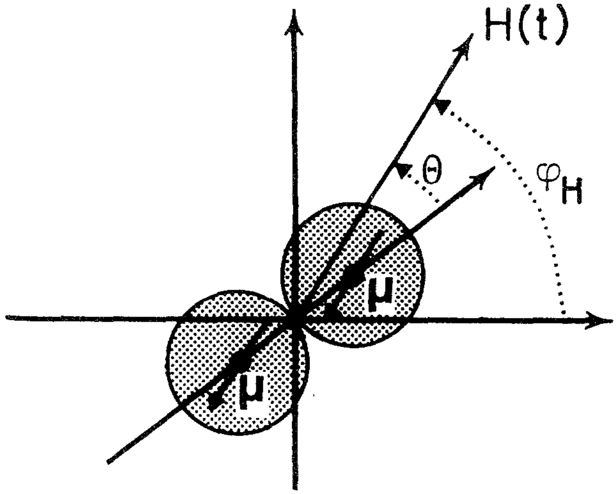

Fig. 1. Schematic diagram of the motion of a magnetic dipole pair due to an elliptically polarized field $\mathbf{H}(t)$.

This is a Riccati equation, which can be transformed to a second-order homogeneous linear equation (Kamke, 1943; Watson, 1944) with the transformation

$$v=\exp\left[-\int u(\tau)\,d\tau\right], \qquad \text{i.e.,} \quad u=-\frac{v'}{v} \tag{10}$$

and then v satisfies the linear equation

$$v''+\frac{2\omega_c}{r}\left[\frac{1+\tan^2\tau}{1+(1/r^2)\tan^2\tau}\right]^2 v'+v=0 \tag{11}$$

which can be transformed into a standard form (Kamke, 1943)

$$v''+I(\tau)v=0 \tag{12}$$

where

$$I(\tau)=1-\left(\frac{\omega_c}{r}\right)^2\left[\frac{1+\tan^2\tau}{1+(1/r^2)\tan^2\tau}\right]^4-\frac{\omega_c}{r}\frac{d}{d\tau}\left[\frac{1+\tan^2\tau}{1+(1/r^2)\tan^2\tau}\right]^2 \tag{13}$$

It is easy to see that $I(\tau)$ is a real function of τ with period π and this is an extended Hill equation. Following Floquet's theory (Kamke, 1943; Wang and Guo, 1965; Whittaker and Watson, 1927), Yang and Yao (1990) solved the extended Hill equation $d^2u(z)/dz^2+J(z)u(z)=0$, where $J(z)$ is a real periodic function of z.

For real values of τ, $I(\tau)$ can be expanded in the form

$$I(\tau)=\sum_{n=-\infty}^{\infty} a_{2n}\exp(i2n\tau) \tag{14}$$

where

$$a_{2n}=\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} d\tau\, I(\tau)\exp(-2in\tau) \qquad (n=0,\pm1,\pm2,\ldots) \tag{15}$$

That function $I(\tau)$ is a real function of τ yields

$$a_{-2n}=a_{2n}^{*} \qquad (n=0,1,2,\ldots) \tag{16}$$

Directly calculating the integrals, one may obtain

$$a_0=1-\frac{\omega_c^2}{16r}(5+3r^2+3r^4+5r^6)$$

$$a_{2n}=-\frac{\omega_c^2}{16r}c_n\left(\frac{1-r}{1+r}\right)^n+in\omega_c(1+2nr+r^2)\left(\frac{1-r}{1+r}\right)^n \tag{17}$$

for $n=1,2,3,\ldots$, where

$$c_n=5+10nr+(3+8n^2)r^2+\tfrac{4}{3}n(2n^2+7)r^3+(3+8n^2)r^4+10nr^5+5r^6 \tag{18}$$

for $n=1,2,3,\ldots$.

We assume the solution ot equation (12) can be written

$$v(\tau)=\exp(i\mu\tau)\sum_{n=-\infty}^{\infty} b_n\exp(in\tau)=\sum_{n=-\infty}^{\infty} b_n\exp(i\mu\tau+in\tau) \tag{19}$$

On substitution in equation (12), we have

$$-b_n(\mu+n)^2+\sum_{m=-\infty}^{\infty} b_{n-2m}a_{2m}=0 \qquad (n=0,\pm1,\pm2,\ldots) \tag{20}$$

By dividing the linear equations of (20) by $a_0-(\mu+2n)^2$, one may obtain the equation system

$$\sum_{m=-\infty}^{\infty} B_{nm}b_{2m}=0 \qquad (n=0,\pm1,\pm2,\ldots) \tag{21}$$

where B_{nm} are coefficients defined by

$$B_{nm}=\begin{cases} 1 & n=m \\ \dfrac{a_{2(n-m)}}{a_0-(\mu+2n)^2}, & n\neq m, \end{cases} \qquad (n,m=0,\pm1,\pm2,\ldots) \tag{22}$$

The μ in equation (19) is determined by the roots of the determinantal equation

$$\sin^2\left(\frac{\mu\pi}{2}\right) = \Delta(0)\sin^2\left(\frac{\sqrt{a_0}\,\pi}{2}\right) \tag{23}$$

where $\Delta(\mu)$ is the extended Hill determinantal equation defined by

$$\Delta(\mu) = \begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 1 & \frac{a_2^*}{a_0-(\mu-4)^2} & \frac{a_4^*}{a_0-(\mu-4)^2} & \frac{a_6^*}{a_0-(\mu-4)^2} & \frac{a_8^*}{a_0-(\mu-4)^2} & \cdots \\ \cdots & \frac{a_2}{a_0-(\mu-2)^2} & 1 & \frac{a_2^*}{a_0-(\mu-2)^2} & \frac{a_4^*}{a_0-(\mu-2)^2} & \frac{a_6^*}{a_0-(\mu-2)^2} & \cdots \\ \cdots & \frac{a_4}{a_0-\mu^2} & \frac{a_2}{a_0-\mu^2} & 1 & \frac{a_2^*}{a_0-\mu^2} & \frac{a_4^*}{a_0-\mu^2} & \cdots \\ \cdots & \frac{a_6}{a_0-(\mu+2)^2} & \frac{a_4}{a_0-(\mu+2)^2} & \frac{a_2}{a_0-(\mu+2)^2} & 1 & \frac{a_2^*}{a_0-(\mu+2)^2} & \cdots \\ \cdots & \frac{a_8}{a_0-(\mu+4)^2} & \frac{a_6}{a_0-(\mu+4)^2} & \frac{a_4}{a_0-(\mu+4)^2} & \frac{a_2}{a_0-(\mu+4)^2} & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0 \tag{24}$$

When $r \sim 1$, with the approximation used in Wang and Guo (1965), one may obtain

$$\Delta(0) \simeq 1 + \frac{\pi\cot(\pi\sqrt{a_0}/2)}{4\sqrt{a_0}}\left(\frac{|a_2|^2}{1^2-a_0} + \frac{|a_4|^2}{2^2-a_0} + \frac{|a_6|^2}{3^2-a_0} + \cdots\right) \tag{25}$$

All the coefficients $\{b_n\}$ can in principle be obtained by solving the equation system (20) with μ given by equations (23) and (25). One may obtain the Floquet solutions to (12) by substituting $\{b_n\}$ and μ into (19).

In summary, we have introduced a time transformation to study problems with elliptically polarized fields, and solved the dynamical equation of a dipole pair in an external elliptically polarized field with this transformation.

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